On a mean field game approach modeling congestion and aversion in pedestrian crowds

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Abstract

In this paper we present a new class of pedestrian crowd models based on the mean field games theory introduced by Lasry and Lions in 2006. This macroscopic approach is based on a microscopic model, that considers smart pedestrians who rationally interact and anticipate the future. This leads to a forward-backward structure in time. We focus on two-population interactions and validate the modeling with simple examples such as self-organization behavior as for instance lane formation. Two complementary classes of problems are addressed, namely the case of crowd aversion and the one of congestion. In both cases we describe the model, build a numerical solver (respectively based on optimization formulation and partial differential equations), and finally provide some numerical tests involving complex group behaviors such as symmetry breaking and lane formation.

Keywords: Mean field games, interacting populations, Nash equilibrium, rational expectations, flow of pedestrians, lane formation, numerical approximation

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1. Introduction

In the last decades crowd motion modeling has become an active area of research. As the world population and the urbanization continues to increase we can not underestimate the importance of understanding the behavior of human crowds. Nowadays there is lot of interdisciplinary research among physicists, sociologists, biologists and mathematicians; crowd motion has become one of the emerging research topics.
The first empirical studies on crowd motion have been deducted about 50 years ago (cf. Hankin and Wright (1958); Hoel (1968)), nowadays various modeling approaches can be found in the literature. We distinguish between two different approaches, either microscopic models or macroscopic ones. The most popular microscopic approaches include the behavioral force models (cf. Helbing (1998, 1991); Helbing et al. (2002); Helbing and Molnar (1995)), cellular automata (cf. Burstedde et al. (2001); Kirchner and Schadschneider (2002)) and stochastic models. Behavioral force models have been very successful in describing various crowd motion phenomena like lane formation, oscillations at bottlenecks or clogging. The main drawback of microscopic approaches is the high computational cost for large number of particles. In macroscopic models the crowd is treated as a density. Here the computational cost is much smaller, but it's less clear how to describe the motion of the density correctly. Several macroscopic models have been derived by considering the limit as the number of particles tends to infinity, e.g. Burger et al. (submitted). Other macroscopic approaches are based on kinetic equations or fluid dynamics

(see e.g. Henderson (1974)). A different macroscopic approach has been proposed by Hughes (2002), Francesco et al. (2010) studied the analysis and numerics in the one-dimensional case. More recent models are based on optimal transportation, see Maury et al. (to appear). For a general overview on the various modeling approaches we refer to Helbing et al. (2002).
Furthermore we would like to point out the connection between human crowd motion models and herding models for animals, like ants, sheep or fish. A model for describing the collective motion of ants has been used successfully to describe lane formation in crowds. Ants use chemical substrates to lead others to valuable food resources. Schadschneider et al. (2007) used a cellular automata model which uses a virtual trace that features collective behavior of humans an subsequent lane formation.

In this paper we would like to present a new approach to model crowd motions. It is called mean field games (MFG) and has been introduced initially by Lasry and Lions, see Lasry and Lions (2007). A particular appealing feature is that the approximating macroscopic mean field model is derived from a mean-field game as the number of players tends to infinity. Consequently it offers the tractability of a macroscopic framework together with the more realistic interpretation at the microscopic level. Furthermore we treat players as "smart" individuals which try to optimize their path with respect to a particular goal. Thus, a MFG model is quite novel for crowd modeling and differentiates itself from the models cited above in mainly two points. First we don't talk anymore about neither particles nor robots nor automata, but rather about real individuals or agents, having strategical interactions within the crowd. Second, these individuals anticipate the future. This is mathematically expressed through a forward-backward structure, the forward dynamic describing the crowd dynamic whereas the backward one is needed to build the expectations. This differs from pure forward frameworks that might be unable to describe particular features of crowd dynamics.
We believe that a MFG model is interesting in the sense that beyond offering a description of how pedestrian behave in a crowd, it also answers the questions why and how the crowd moves (or why it has its specific shape in the stationary case). Here is an example of why it could be useful: if for any exogenous reason some people in the crowd suddenly stop for a while, the collective behavior will be re-established. This cannot happen in automata models.
As indicated in the name mean field games, it is a game approach and has the very important feature to consider the pedestrians' behavior as a non-cooperative equilibrium. Therefore it is unfortunately not possible to consider cases of small groups interacting within the crowd.

We reiterate that the limiting MFG are motivated by an N-player stochastic game. These limiting macroscopic partial differential equations (PDEs) have a surprisingly simple structure, which allow the development of efficient numerical methods. Different numerical approaches for MFG can be found in the literature (see for instance Achdou and Capuzzo-Dolcetta (2009); Guéant (2009); Lachapelle (2010); Lachapelle et al. (2010); Lions (2008-2010)). They are either based on the optimal control formulation of the problem or on the PDE system. In the first case different methods from parabolic optimal control theory are available. On the PDE level Newton-type methods have been used successfully for numerical simulations. Nevertheless the forward-backward structure and the highly nonlinear equations still pose a great challenge to numerical analysts and the development of efficient numerical methods is still in its early stages.

This paper is organized as follows. Section 2 is devoted to the general introduction of mean field games in a crowd dynamic framework. Next we turn to the central topic of our paper, that is the study of two interacting populations (or equivalently crowds). More particularly we present two different classes of MFG (in order to show the diversity of cases that could be treated by MFG). For both of them we describe the modeling, discuss a numerical solver and provide numerical results. In section 3 we present a model where individuals are averse to the crowd, in a time-dependent framework. The gradient-descent numerical method is based on an optimal control formulation of the problem. Our numerical tests mainly provide a symmetry breaking in some xenophobic situation (where the pedestrians of the first group really dislike the pedestrians of the other group). In section 4 we look at a two-species stationary model involving congestion. Here the numerical simulations, based on Newtons' method, reproduce well known phenomena in crowd motion like lane formation. Finally, we close with some concluding remarks in section 5.

2. Mean field games: a micro-macro approach to model pedestrian crowds

In this section we discuss the general concept of mean field games and its application to crowd motion dynamics. For reasons of clarity and readability we start with the introduction of mean field games for single species first. This discussion can easily be generalized for two populations, which is of particular interest when studying interactions among crowds.
First we recall the microscopic fundament of MFG, that is a multi-player stochastic differential games. Then we present the approximation of such games when the number of player tends to infinity (without entering the mathematical details), namely the associated mean field game. Various remarks throughout the section have been added to clarify the modeling assumptions.

2.1. Microscopic fundament: stochastic differential games

Let us start at the microscopic scale of a MFG, namely an N-player stochastic differential game. We look at an evolution game on a period $[0, T]$ in a stochastic framework. Here we choose the probability space $(\Omega, \mathfrak{F}, \mathcal{P})$ where $\mathfrak{F} = (\mathfrak{F}_t)_{t\in[0,T]}$ is the filtration enhanced by a vector (with length N) of d-dimensional independent Brownian motions $W = (W_t)_{t\in[0,T]} = \big((W_t^1)_{t\in[0,T]}, ...(W_t^N)_{t\in[0,T]}\big)$. We consider N players (also referred as pedestrians or individuals) interacting one with each other through the choice of an action (or control). At every time t player i chooses an action α^i from the set of feasible actions $A_i \subset \mathbb{R}^d$, for $i = 1, ..., N$. We define $A = A_1 \times ... \times A_N$ and the global set of admissible actions as

$$\mathcal{A} = \{\alpha = (\alpha_t)_{t\in[0,T]}; \alpha \text{ is progressively measurable in } A\}.$$

Consequently, $(\alpha_t^i)_{t\in[0,T]}$ denotes the strategy or control process of player i.
For any $\alpha = (\alpha^1, ..., \alpha^N)$ we will use the very convenient and usual notation in game theory $(\alpha_i, \alpha^{-i}) = (\alpha^1, ..., \alpha^{i-1}, \alpha_i, \alpha^{i+1}, ..., \alpha^N)$.
We are interested in simple cases where the state evolution of the system in $\mathbb{R}^{N\times d}$ is given by the solution of the following Stochastic Differential Equation (SDE for short):

$$dX_t = \alpha_t dt + \sigma dW_t,\ X_0 = x \in \mathbb{R}^{N\times d}, \tag{1}$$

where σ is a $Nd \times Nd$ diagonal matrix (independence case).

Remark 1. *In other words player i evolves controlling the drift of the controlled stochastic process $dX_t^i = \alpha_t^i dt + \sigma dW_t^i$, starting at x^i. Note that we will focus on the realistic particular case where the crowd evolves in a bounded subset of $\mathbb{R}^2$ (in this case we will specify the boundary conditions later). We reiterate that the speed of each individual is not fixed a priori but rather*

determined by each pedestrian as the result of an optimization process (it is the control α). We come back on this comparison later, when we will describe the macroscopic setting.
We emphasize that pedestrians are subject to independent noise. If we would consider correlated noises, the resulting mean field game would be much more complicated, see e.g. Lions (2008-2010).

It is well-known that for every $\alpha \in \mathcal{A}$ there exists a unique solution $X_t = X_t^\alpha$ to (1). Next we discuss how player i determines its optimal strategy by a cost functional. This cost functional consists of two parts, the first one corresponds to the running cost $f_i : [0,T] \times \Omega \times \mathbb{R}^{N\times d} \times A_i \to \mathbb{R}$ and the second one to the terminal cost $g_i : \Omega \times \mathbb{R}^{N\times d} \to \mathbb{R}$. Then for each $\alpha \in \mathcal{A}$ we define the cost functional

$$J_i(\alpha) = \mathbb{E}\left[\int_0^T f_i(t, X_t^\alpha, \alpha_t^i)dt + g_i(X_T^\alpha)\right]. \tag{2}$$

Note that we will also study the stationary case for which we will consider a slightly different definition of the cost functional namely:

$$J_i^s(\alpha) = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T f_i(t, X_t^\alpha, \alpha_t^i)dt\right]. \tag{3}$$

Next we discuss the special case of mean-field type interactions between players i.e. we discuss special forms of costs and their very important aspect in the modeling.
Each pedestrian takes into account her own state X_t^i and the mean-field created by the others, i.e. the empirical distribution of pedestrians $\hat{x}^{-i} = \frac{1}{N-1}\sum_{j\neq i}\delta_{x^j}$ where δ_x denotes the Dirac mass located at x. Then for each i and $x = (x^1, ..., x^N)$, the running and terminal costs have the particular form $f_i(t, x, \alpha) = f(t, x^i, \hat{x}^{-i}, \alpha)$ and $g_i(x) = g(x^i, \hat{x}^{-i})$ and we have

$$J_i(\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t^{i,\alpha}, \hat{X}_t^{-i,\alpha}, \alpha_t^i)dt + g(X_T^{i,\alpha}, \hat{X}_t^{-i,\alpha})\right].$$

Remark 2. *Here we make the implicit assumptions that each player is anonymous. Indeed they are identical but can be different (that is we cannot identify who's who in the crowd but everyone can have her own path that may differ from the others). This can be seen through the form of the costs. For a fixed index i, J_i is invariant by any permutation of the $-i$ other players.*
In our model we assume that the pedestrians are rational, with rational expectations, i.e. every individual or agent knows and anticipates the state of everybody through the distribution of the crowd. These assumptions are probably wrong in the real world (except in routine crowd situations, say the daily way towards the subway exit). Indeed the pedestrians could be partially and insufficiently informed, or some irrational local behavior can happen (e.g. an individual suddenly stops to observe the crowd). But as it is the case in physics, we first have to introduce the model with rational expectations and rational individuals in order to be able to correct and improve the model afterwards (e.g. by introducing partial blindness or an extra noise for instance). A robot model might be more intuitive at the first glance, but it might fail if the environment changes.
A first step to consider partial blindness can be performed either through the introduction of a discount factor (chosen significantly high) which has been done in section 4 or considering running costs involving a convolution. The convolution of the empirical density with the indicator function of a circle of radius r corresponds to taking actions of individuals in certain neighborhood into account. Indeed we do not enter the details of the dependency of the cost upon the

crowd empirical distribution, we do not even specified whether it is local or non-local. Here we will only consider fully non-local dependencies, the generalization to other cost functions will be subject of further research.

We reiterate that mean field games are an equilibrium-based approach. Hence we try to find a Nash equilibrium (or Nash point). A Nash point is a situation in which every player cannot improve his payoff by any unilateral deviation. That is a strategy $\overline{\alpha}$ such that

$$\forall i \in \{1, ...N\}, \forall \alpha_i \in \mathcal{A}_i, J_i(\alpha_i, \overline{\alpha}^{-i}) \geq J_i(\overline{\alpha}).$$

Existence of Nash equilibria can be shown under very general assumptions on the cost. We will only consider quadratic dependence of f_i upon α^i, which is covered by standard existence results. A rigorous study of such problems can be found in Bensoussan and Frehse (1984). Furthermore they show that the differential game is equivalent to a system of coupled PDEs, that are intractable when N is large. Next we discuss an approximation of such games by a mean field game with a continuous distribution of players.

2.2. Macroscopic framework: Mean Field Game equations

Though the notion of games with a continuum of players is not novel (see the work of Aumann (1964)), it has not been adapted to stochastic games (the difficulty comes from the independence of white noises when taking $N \to \infty$). Our aim is not to enter the mathematical details of taking limits when $N \to \infty$, we rather give a common intuition of how to derive the MFG system in the continuum of player setting. For more details on the highly technical proof of the approximation result, we refer to Lasry and Lions (2007) and Lions (2008-2010).
Therefore we directly jump to the continuum of players setting. First we present the equations of the MFG system (whose solutions are mean-field equilibria or equivalently approximation of Nash equilibria) both in the dynamical and stationary cases. Then we briefly motivate a common intuition on how to derive MFG equations.

2.2.1. Dynamical and stationary MFG systems

We distinguish between finite horizon MFG and stationary MFG. This is motivated by the two differential crowd motion problems which we study in this paper. In section 3 we look at crowd motion dynamics and therefore consider the following (and more general) type of finite horizon MFG system:

$$\partial_t v + \frac{\sigma^2}{2}\Delta v + H(t, x, \nabla v, m) = 0, \ v|_{t=T} = g(m_T) \tag{4}$$

$$\partial_t m - \frac{\sigma^2}{2}\Delta m + \mathrm{div}(m\ \partial_p H(t, x, \nabla v, m)) = 0, \ m|_{t=0} = m_0. \tag{5}$$

The mathematical structure of this system captures many features of MFG modeling. It is a forward-backward system coupling two PDEs. Equation (4) is a Hamilton-Jacobi-Bellman (HJB) equation on the adjoint variable v. It is related to the control aspects and basically gives the optimal strategy obtained by a backward reasoning (feedback). Note that there are two sources of coupling in this equation: both in the Hamiltonian H and in the terminal condition of the adjoint variable v. Here the Hamiltonian H is defined as the Legendre transform of the running cost f, i.e. $H(t, x, p, m) = f^*(t, x, p, m) := \sup_\alpha\{p\alpha - f(t, x, \alpha, m)\}$.
On the other hand, equation (5) describes the mass evolution equation of the system. It is called

a Kolmogorov equation in financial mathematics or a Fokker-Planck equation in physics. Here m_0 denotes the initial pedestrian distribution. Its solution m_t is the distribution of pedestrians transported according to individuals' optimal choices (forward in time). The optimal strategy is given by $\alpha = \partial_p H(., \nabla v, m)$.
The forward-backward structure is a core aspect of MFG. The achievement of a MFG equilibrium (i.e. a solution of the system (4)-(5)) is performed through the rational expectations assumption (we come back more in details on this very important point in the next subsection).

In section 4 we look at the corresponding stationary case that is a MFG system having the following form:

$$\frac{\sigma^2}{2}\Delta v + H(x, \nabla v, m) - rv + \lambda = 0 \tag{6}$$

$$-\quad \frac{\sigma^2}{2}\Delta m + \operatorname{div}(m\,\partial_p H(x, \nabla v, m)) = 0 \tag{7}$$

$$\lambda \in \mathbb{R},\ \int m dx = 1,\ m > 0,\ \int v dx = 0. \tag{8}$$

The structure of the stationary problem is very similar to the the time-dependent one. Note the additional variable λ, which gives the optimal value of the equivalent of (3) in the game with a continuum of players, and the distribution's conditions in (8). The linear term rv appearing in (6) comes from the introduction of a discount factor r in the cost functional. We reiterate that the rigorous derivation of these PDEs systems can be found in Lasry and Lions (2007) and Lions (2008-2010).

2.2.2. Formal derivation of the finite horizon system

In order to better understand the forward-backward structure, we would like to give an intuition on the derivation of MFG. We consider an equivalent formulation of the individual problem consisting of minimizing (2) in the infinite number of players game. Here the starting point is the individual problem of a single individual located at x:

$$\begin{cases} \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^T f(t, X_t^{\alpha,x}, \alpha_t, m_t)dt + g[m_T](X_T^{\alpha,x})\right] \\ dX_t^{\alpha,x} = \alpha_t dt + \sigma dW_t\ ,\ X_0^x = x. \end{cases} \tag{9}$$

Note that we replace the empirical distribution $\frac{1}{N-1}\sum_{j\neq i}\delta_{X_t^j}$ by m_t. Each individual in the continuum takes into account the global distribution evolution $(m_t)_{t\in[0,T]}$, hence anticipating the future. We proceed in two steps in order to get the system (4)-(5). First we assume that $(m_t)_{t\in[0,T]}$ is known (expectations hypothesis) and solve the stochastic control problem. We introduce the value function:

$$v(t, x) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_t^T f(s, X_s^{\alpha,x}, \alpha_s, m_s)ds + g[m_T](X_T^{\alpha,x})\right].$$

It is well known that the value function v is a solution to the HJB equation (see Bensoussan and Frehse (1984))

$$\partial_t v + \frac{\sigma^2}{2}\Delta v + H(t, x, \nabla v, m) = 0\ ,\ v|_{t=T} = g[m_T].$$

We reiterate that we define the optimal feedback as a maximizer in the previous formula, computed at the point $(t, x, \nabla v(t, x), m_t(x))$, i.e. $\alpha(t, x, m) = \partial_p H(t, x, \nabla v(t, x), m_t(x))$.
The Kolmogorov equation describes the evolution of m_t, it is a transport by drift (here α) and diffusion equation. For each $t \in [0, T]$, define the distribution as follows:

$$\int \varphi dm_t = \int \mathbb{E}\left[\varphi(X_t^{\alpha,x})\right] dm_0(x) \text{ for all test function } \varphi.$$

Then the Kolmogorov equation follows from applying Itô's Lemma

$$d\varphi(X_t^x) = [\alpha(t, X_t^x, m_t)\nabla\varphi(X_t^x) + \frac{\sigma^2}{2}\Delta\varphi(X_t^x)]dt + \sigma\nabla\varphi(X_t^x)dW_t.$$

Indeed, by a simple integration by parts one can write

$$\begin{aligned}
\int_0^T \varphi\partial_t m &= \frac{d}{dt}\int_0^T \varphi dm_t = \int_0^T \mathbb{E}[d\varphi(X_t^x)]dm_0(x) \\
&= \int_0^T \mathbb{E}[\alpha(t, X_t^x)\nabla\varphi(X_t^x) + \frac{\sigma^2}{2}\Delta\varphi(X_t^x)]dm_0(x) \\
&= \int_0^T \varphi[-div(m\alpha) + \frac{\sigma^2}{2}\Delta m],
\end{aligned}$$

and we finally obtain the desired equation.
Now we are able to put everything together. For $\alpha(t, x) = \partial_p H(x, \nabla v(t, x), m_t(x))$ we obtain the Kolmogorov equation:

$$\partial_t m - \frac{\sigma^2}{2}\Delta m + \text{div}(m\ \partial_p H(t, x, \nabla v, m)) = 0,\ m|_{t=0} = m_0.$$

Remark 3. *The forward-backward aspect can be considered as the heart of the MFG theory and refers to the rational expectations equilibrium approach. Indeed individuals anticipate the crowd evolution first and then evaluate their cost function (9). Next they deduce their strategy (feedback control) by solving the HJB equation. Finally the mass evolves according to these strategies (Kolmogorov equation). At the optimum the mass evolution has to coincide with the one which has been anticipated, according to the rational expectations assumption.*
Even in very simple configurations, a pure forward model should not be able to predict the crowd dynamic. This is why we believe that it is necessary to introduce forward-backward models such as MFG. Indeed, if we want to find expectation phenomenon, it is necessary to have both a forward equation (to describe the real dynamic) and a backward equation (to build up the expectations).

2.2.3. Link with optimal control

Finally we would like to complete the picture by linking mean field games to optimal control problems. MFGs are equivalent to optimal control problem under certain assumptions on the cost function. One of the presented crowd motion models is equivalent to an optimal control problem, the other not. In the first model, namely the crowd aversion model of section 3, we take full advantage of the optimal control structure in the numerical discretization. However, this won't be possible in the second model of section 4, where we take into account congestion.

An optimal control problem is equivalent to a MFG if the following criteria are satisfied. First the running cost can be written $f(t, x, \alpha, m) = L(t, x, \alpha)m + V[x, m]$ and second V and g are the derivatives (e.g. the Gâteaux derivatives) of two potentials Φ and Ψ on bounded measures, i.e. $V = \Phi'$ and $g = \Psi'$. In this case, the critical points of the optimal control problem of Kolmogorov equation

$$\begin{cases} \inf_{\alpha\in A} J(\alpha) := \int_0^T \int_\Omega L(x, \alpha)m(t, x)dx + \Phi(m)(t)dt + \Psi(m_T) \\ \partial_t m - \frac{\sigma^2}{2}\Delta m + \mathrm{div}(\alpha m) = 0 \ , \ m(0, .) = m_0(.), \end{cases} \tag{10}$$

are solutions of the MFG system (4)-(5). A proof of this statement can be found in Lachapelle (2010). Note that this is a sufficient condition as soon as Φ and Ψ are convex. Consequently, we see that the continuum of individual problems can be seen as a global optimization (or control) problem.
Finally we present some concluding remarks on our modeling approach.

Remark 4. *First, it is clear that when we use the terminology of game we refer to classical game theory with an equilibrium approach and not to evolutionary game theory used in biology (also for crowds modeling and socio-behavioral modeling, see Haag and Weidlich (1983) or Helbing (1998)).*

- *We reiterate one more time the difference with social force models (see Helbing (1991); Helbing and Molnar (1995)) where pedestrian acts like automata (even with a memory criterium). Here this can be typically understood by looking at this forward-backward structure and the term in the divergence of the Kolmogorov equation (5), which is not fixed a priori since it is the feedback control.*

- *Moreover, the non-cooperative equilibrium approach is a bit restrictive since it does not allow to model more realistic pedestrian crowds with small social groups therein (2 or 3 people groups for instance).*

- *Finally a satisfying point we want to mention is that contrary to other game models, it allows entries and exits of pedestrians in the crowd (see section 4 for an example). Indeed, in games with a finite number N of players, it is very complicated to consider a non-constant N.*

In the next sections, we illustrate how MFG can be used for modeling pedestrian crowds. More precisely we introduce two different models for 2-population interactions, the first one is based on crowd aversion while the second one takes into account congestion in the running cost. We present numerical discretizations and simulations in both cases.

3. Pedestrian dynamics in the case of xenophobia: symmetry breaking

In this part we model two groups dynamics, each of whom consisting of infinitely many pedestrians. Consequently we use the finite horizon setting. We will focus here on a MFG that can be written as an optimal control problem (the running cost has the form $f(t, x, \alpha, m) = L(t, x, \alpha) + V(x, m(x))$ as mentioned in section 2. The optimal control formulation serves as a starting point for our numerical discretization, namely a gradient descent method. Note that we will only use the macroscopic optimal control formulation, but nevertheless, it is good to keep in mind the microscopic basis.
The basic modeling assumption is that pedestrians in both groups have an aversion towards

crowds. Furthermore the groups want to avoid each other (xenophobia). This approach differs from the one we use in the next section where the pedestrians' choices take congestion into account (that is the cost to move increases with the crowd distribution). Here crowd aversion only consists in penalizing congestion, not modeling it. The first of the previous aspects is mathematically expressed as convexity in the group problem while we will see that the latter leads to the loss of convexity in the joint problem for both groups.

We start with the macro-Nash problem between two populations (each of them being MFG), then state some theoretical results, detail the numerical algorithm and present some simulations including symmetry breaking.

3.1. Writing a macro-Nash problem between two groups

Let us focus on the case where two populations interact inside a subset Ω of $\mathbb{R}^2$. We also define the time-space domain $Q = [0,T] \times \Omega$.

We want to study equilibria between the two groups (typically Nash points as suggested by Lasry and Lions (2007)). Formally, the global optimization problem (linked to a continuum of individual problems) of group i, given the control and the mass evolution of the other group (i.e. $(\alpha_t^{-i}, m_t^{-i})$), reads as:

$$\inf_{\alpha^i} J^i_\lambda(\alpha)$$

where

$$J^i_\lambda(\alpha) = \int_Q \frac{|\alpha^i|^2}{2} m^i + \int_0^T \Phi^i_\lambda(m^1_t, m^2_t) + \int_\Omega \Psi^i m^i(T). \tag{11}$$

Note that the upper index now refers to the group (or population) and not to the individual player. From now on, m^i depends upon α^i, more precisely it is viewed as a bounded nonnegative measure (i.e. belonging to the set $\mathcal{M}_b(Q, \mathbb{R}_+)$) which is a weak solution of the Kolmogorov equation:

$$\partial_t m^i - \frac{\sigma^2}{2} \Delta m^i = -\mathrm{div}(\alpha^i m^i), m^i(0) = m^i_0. \tag{12}$$

We distinguish the populations by considering different initial densities $m^i_0(.)$ and different final incentive costs Ψ^i, for $i = 1, 2$. However we study the simple case where the Brownian motion and the noise are similar for both groups. In the definition of the criterion (11), the coupling cost we consider is

$$\Phi^i_\lambda(m_t) := \int_\Omega (m^i_t)^2 + \lambda m^1_t m^2_t,$$

for a nonnegative real constant λ. This models situations in which individuals have an aversion towards members of his/her own group as well as the other group. Therefore λ can be interpreted as the *"xenophobia"* parameter. To see that, note that the individual mean field criterion is:

$$V_i[m_t](x) = 2m^i_t(x) + \lambda m^{-i}_t(x).$$

In fact we consider that xenophobia is significant when the aversion λ to the other group is higher than the one to the own group (equal to 2). The Nash problem between the groups is then:

$$(\mathcal{N}) \text{ Find } \overline{\alpha} = (\overline{\alpha}^1, \overline{\alpha}^2) \text{ such that: } J^i_\lambda(\overline{\alpha}) = \inf_{\alpha^i \in \mathcal{M}_b(Q, \mathbb{R}^d)} J^i_\lambda(\alpha^i, \overline{\alpha}^{-i}), \text{ for } i = 1, 2.$$

3.2. Existence and optimality

3.2.1. Optimality conditions

In this part we present a characterization of the Nash equilibria. To do so, let us introduce the MFG system for two groups: for $i = 1, 2$,

$$\partial_t m^i - \frac{\sigma^2}{2}\Delta m^i + \mathrm{div}(m^i \nabla v^i) = 0,\ m^i(0) = m_0^i, \tag{13}$$

$$\partial_t v^i + \frac{\sigma^2}{2}\Delta v^i + \frac{|\nabla v^i|^2}{2} = \Phi_\lambda^i(m)',\ v^i(T) = \Psi^i, \tag{14}$$

and the joint minimization problem (for the two groups)

$$(Q)\ \inf_{\alpha=(\alpha^1,\alpha^2)} J_\lambda(\alpha) := J^1_{\lambda/2}(\alpha) + J^2_{\lambda/2}(\alpha),$$

under the constraints: m_i is a solution of (12), for $i = 1, 2$. Note that J_λ is convex if the xenophobia parameter is small, that is $\lambda \leq 2$. Now we are able to state the following optimality conditions.

Proposition 3.1. *If $\lambda \leq 2$ then the following assertions are equivalent:*

1. $\overline{\alpha} \in \mathcal{M}_b(Q, \mathbb{R}^d)$ *is a solution of* $(\mathcal{N})$ *and* $\overline{m}$ *satisfies (12) for* $\alpha = \overline{\alpha}$,
2. $\overline{\alpha} \in \mathcal{M}_b(Q, \mathbb{R}^d)$ *is a solution of* (Q) *and* $\overline{m}$ *satisfies (12) for* $\alpha = \overline{\alpha}$,
3. $(\overline{m}, \overline{v})$ *is a solution of the MFG system (13)-(14), with* $\alpha = \overline{\alpha} = \nabla\overline{v}$.

If $\lambda > 2$ then it is only necessary i.e. 2. $\Rightarrow$ 1.,3.

A proof of this statement is based on classical arguments of differential calculus and can be found in Lachapelle (2010). It basically consists of introducing first order perturbations, the linearized PDE and compute the first variation.

3.2.2. Existence

If problem (Q) has a solution, then proposition 3.1 ensures the existence of a Nash point between the two groups. In order to give an existence result, we may reformulate the problem following Benamou and Brenier (2000); Buttazzo et al. (2009); Lachapelle et al. (2010). We adopt a vectorial point of view, and use the following notations:

- $m = (m^1, m^2)$, and for all $x = (x^1, x^2) \in \mathbb{R}^2_{+*}$, $\frac{1}{x} := (\frac{1}{x^1}, \frac{1}{x^2})$,
- $q = (q^1, q^2)$, and for all $y = (y^1, y^2) \in \mathbb{R}^d \times \mathbb{R}^d$, $|y|^2 := (|y^1|^2, |y^2|^2)$,
- $\mathcal{A} := \{(q, m) :\ \int_Q (\partial_t u + \frac{\sigma^2}{2}\Delta u) dm^i + \int_Q \nabla u.dq^i = \int_\Omega \Psi^i m^i(T) - u_0 m_0^i\ ,\ \forall u \in C^\infty(Q),\ i = 1, 2\}$.

Next we formally introduce the new variables $q = (\alpha m, m)$ and the corresponding cost functional

$$K(q, m) = \begin{cases} J_\lambda(\alpha), \text{ if } q << m \text{ and } q = \alpha m \\ +\infty \text{ else,} \end{cases}$$

where $q << m$ means that the measure q is absolutely continuous with respect to m. In this setting, a rigorous formulation of (Q) is

$$\inf_{(q,m)\in\mathcal{A}} K(q, m). \tag{15}$$

The reformulation allows us to prove the following proposition.

Proposition 3.2. *If $\lambda \le 2$ and $m_0^1,\ m_0^2 \in L^2(Q)$, then problem $(\mathcal{Q})$ possesses a solution (which is unique as soon as $\lambda < 2$). Moreover there exists a Nash point i.e. a solution of $(\mathcal{N})$.*

Remark 5. *Note that existence does not fail when adding a constraint of the type $m \le constant$ as in Maury et al. (to appear).*

Given the reformulation (15), the proof is a simple adaptation of the one obtained by Buttazzo et al. (2009). The key points are to work on the space of bounded measures, to prove that the functional K is lower-semicontinuous for the weak-$\star$ topology and use Jensen's inequality. A complete proof of this proposition is provided in Lachapelle (2010). In the next section we deal with defining a numerical procedure to approximate the solution.

3.3. Numerical setting

In this part we introduce the discretization and a gradient descent method in order to approximate the solution(s) of problem $(\mathcal{N})$. More precisely, we distinguish the cases when the joint problem $(\mathcal{Q})$ is convex from when it is not. In the convex setting (i.e. when $\lambda \le 2$) we describe the gradient descent that we apply to the joint functional. The non-convex case $\lambda > 2$ (in which the xenophobia is significant) is more involved but interesting (we expect non-uniqueness). Here we use an alternating directions method which takes advantage of the convexity of group i's problem, given group $(-i)$'s evolution.

Gradient First of all, let us write the gradient formula of the functional. We look at the reformulated problem given by (15). We slightly modify the point of view considering that the density m is an affine function of the momentum q. To fix ideas, the joint problem reads as:

$$\inf_q F(q) := \sum_{i=1,2}\left(\int_Q \frac{|q_t^i|^2}{2m_t^i} + \Phi^i_{\lambda/2}(m_t) + \int_\Omega \Psi^i m_T^i\right), \tag{16}$$

where m^i, $i = 1,2$ solves :

$$\partial_t m^i - \frac{\sigma^2}{2}\Delta m^i = -\mathrm{div}(q^i),\ \ m^i(0,.) = m_0^i(.). \tag{17}$$

The gradient of the functional can be calculated using classical differential calculus,

$$\forall (q,m) \in \mathcal{A}, \forall w = (w^1,w^2) \in \mathcal{M}_b(\mathbb{R}^{2d}),\ \ \nabla F(q).w = \left(\int_Q \Big(\frac{q^i}{m^i} + \nabla\theta^i\Big).dw^i\right)_{i=1,2}, \tag{18}$$

where θ^i satisfies, for $i = 1,2$: $\theta^i|_{t=T} = \Psi^i$ and

$$-\partial_t\theta^i - \frac{\sigma^2}{2}\Delta\theta^i = -\frac{|q^i|^2}{2(m^i)^2} + (2m^i + \lambda m^{-i}). \tag{19}$$

Algorithm for the convex case Since problem (16) is convex if $\lambda \le 2$, we use a gradient descent method. We focus on the 2D-case ($d = 2$) and take $\Omega = [0,1]^2$ with periodic boundary conditions.
Let M and N be two positive integers, we define the time and space steps by $dt = \frac{1}{N}$ and $dx = \frac{1}{M}$. For $(i,j,k) \in A := \{0,...,N\} \times \{0,...,M\}^2$, for a given function f defined on Q, $f^i_{j,k}$ denotes the

numerical approximation of $f(idt, jdx, kdy)$. Equations (17) and (19) are iteratively solved by a finite difference method. We use the following approximations

$$\begin{aligned}\partial_t f(idt, jdx, kdy) &= \frac{f^{i+1}_{j,k} - f^i_{j,k}}{dt},\\ \Delta f(idt, jdx, kdy) &= \frac{f^i_{j+1,k} - 2f^i_{j,k} + f^i_{j-1,k}}{(dx)^2} + \frac{f^i_{j,k+1} - 2f^i_{j,k} + f^i_{j,k-1}}{(dy)^2}.\end{aligned}$$

At step n, let $f^{(n)} := \left(f^{i,(n)}_{j,k}\right)_{(i,j,k)\in A}$. Then the gradient descent method (GDM) reads as:

1. *Initialization:*
   Choose $q^{(0)}$ then compute $m^{(0)}$ by solving (17) with the finite difference scheme.
2. *Step n:*
   - Compute $\theta^{(n)}$ by solving numerically (19) with $q^{(n-1)}$ and $m^{(n-1)}$, then compute the discretized gradient $\nabla F(q^{(n-1)})$ (formula (18)), using $\theta^{(n)}$.
   - Compute the descent: $q^{(n)} = q^{(n-1)} - \rho_n \nabla F(q^{(n-1)})$.
   - If $\|q^{(n)} - q^{(n-1)}\|$ <Tol1, then stop the algorithm (Tol1 is a tolerance threshold defined by the user).
     Else, $n = n + 1$.

Note that ρ_n above is the descent step size, it is chosen optimal, i.e. minimizing the following: $\rho \in [0, 1] \to F(q^{(n-1)} - \rho\nabla F(q^{(n-1)}))$.

Alternating directions method for the non-convex problem We expect non-uniqueness in the case where aversion to the other group is significant ($\lambda > 2$). It is well known that classical gradient descent methods have convergence problems or diverge due to the non-convexity of the functional. Therefore we use the alternating directions method, which uses the convexity of both group i's problem. The algorithm reads as: Given group $(-i)$'s evolution solve

$$\inf_{q^i} F^i(q) := \int_0^T \int_\Omega \frac{|q^i_t|^2}{m^i_t} + \Phi^i_\lambda(m_t)dt + \int_\Omega \Psi^i m^i_T,$$

where m^i solves (17) for q^i, $i = 1, 2$. One can easily get the formula of the gradient of F^i looking at the joint case (18)-(19). Note that we can deduce m from our computation of q. The idea of the alternating directions method is to apply GDM successively for each group. Note that the upper index refers to the group number and the lower one to the iteration.

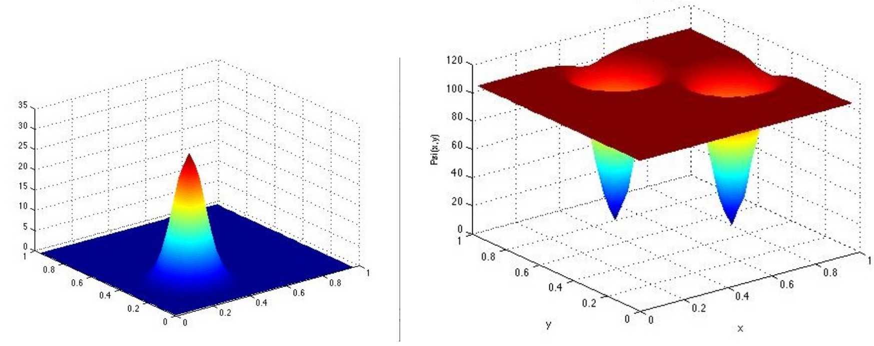
1. *Initialization:*
   Choose q^1_0 then compute q^2_1 with GDM and q^1_0.
2. *Step $k \geqslant 1$:* We know q^2_k.
   - Compute q^1_k then q^2_{k+1} by using GDM (with, respectively, q^2_k and q^1_k).
   - If $\|q^i_k - q^i_{k-1}\|$ <Tol2 for $i = 1, 2$, then stop the procedure.
     Else, $k = k + 1$.

3.4. Simulations

The GDM shows good convergence results when the initials densities of individuals are significantly positive (i.e. $m^i_0 >$ constant > 0). We set $T = 1$ and $\frac{\sigma^2}{2} = 0.01$ in all numerical simulations in this section.

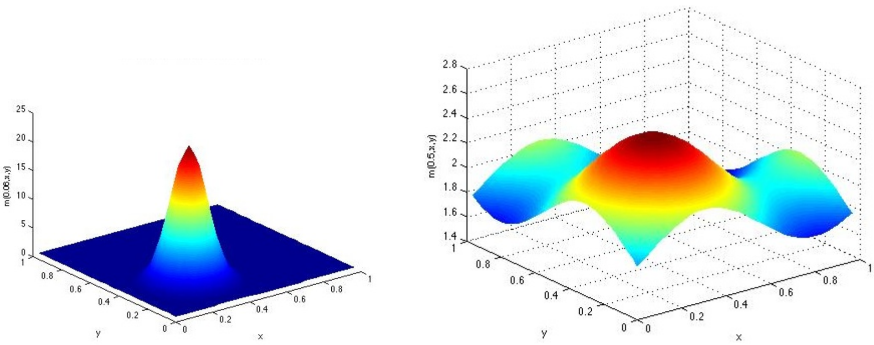
3.4.1. Test 1: crowd aversion in a single group

In the first example we focus on a case involving only one population ($m_0^2 = 0$), i.e. a similar framework as the one studied by Buttazzo et al. (2009). Fig. 1 shows the initial density of agents (centralized around the point $(0.1, 0.1)$) and the final cost, modeling a strong incentive for individuals to be in some neighborhood of $(0.5, 0.8)$ and $(0.8, 0.5)$ at instant T.



(a) Initial density (b) Final cost Ψ

Figure 1: Data



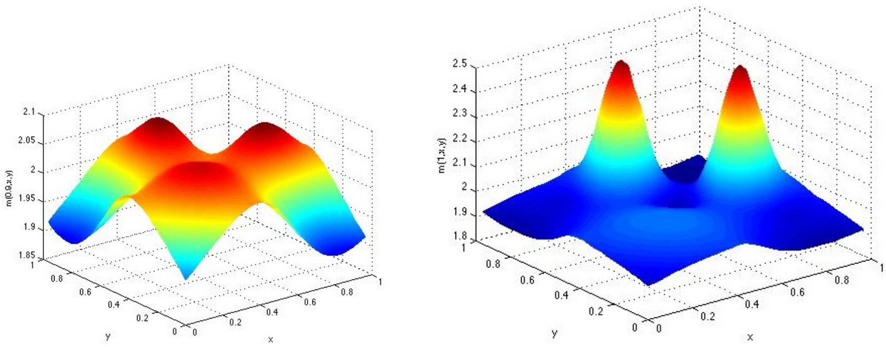
(a) Density at instant $t = 0.06$ (b) Density at instant $t = 0.5$

Figure 2: Spreading over during the first half

Fig. 2 and Fig. 3 present the mass evolution at some chosen instants in $[0, T]$. More precisely, we may observe on Fig. 2 a first step corresponding to a spreading over of m (explained by the aversion term and the diffusion parameter). Note that the running time of dispersion is greater than one half. We then observe in Fig. 3 a split inside the population so that individuals can converge to the two attractive areas. Finally, the discrete energy seems to reach quickly the minimum (5 iterations), see Fig. 4.

3.4.2. Test 2: groups interactions

Next we consider a more interesting example with two populations. Recall that we look for Nash equilibria between two groups whose global optimization problem is (15). We use the



(a) Density at instant $t = 0.9$ (b) Density at instant $T = 1$

Figure 3: Splitting and centralization during the second half

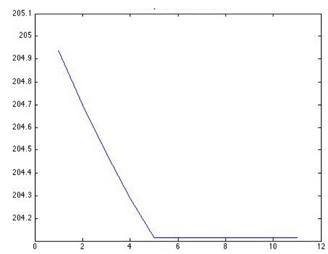
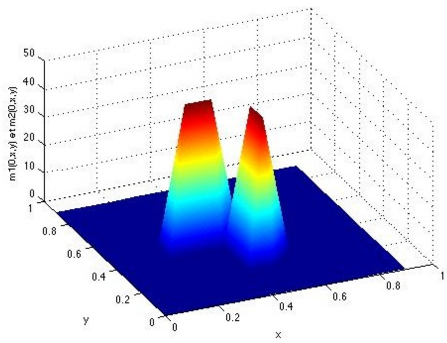
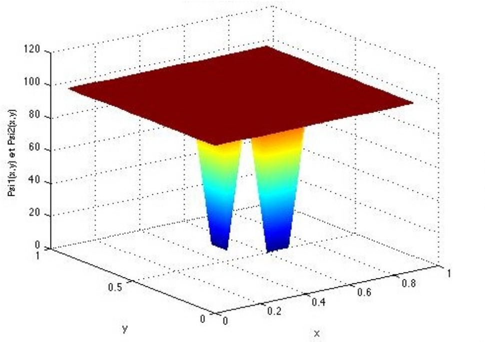


Figure 4: Value of F for each iteration

procedure detailed before (starting with group 2). In order to emphasize the xenophobia behavior we choose $\lambda = 20$ in the definition $\Phi^i_\lambda(m^1, m^2) = \int_\Omega (m^i)^2 + \lambda m^1 m^2$. We consider a symmetric configuration and represent the graphs of the initial densities (m^i_0, $i = 1, 2$) and final costs (Ψ^i, $i = 1, 2$) in Fig. 5. Group 1 is initially centered around $(0.35, 0.5)$, group 2 around $(0.5, 0.35)$. Concerning the final costs the situation is still symmetric since they model incentives to reach positions in the neighborhood of (respectively for group 1 and 2) $(0.65, 0.5)$ and $(0.65, 0.5)$.
With such a situation we are interested in crossing phenomenon. The graphs of both densities are depicted in Fig. 6. Note that we observe the same spreading as in Test 1. However, the most interesting evolution period is described in Fig. 7. Indeed, we can see that group 1 gives the priority to group 2 to go to its attractive area passing through the center of the domain (the shortest road for the euclidian metric). Some of the individuals of group 1 wait, some others go through the border (periodic conditions), and the lasts go through the center (the most congested area). Anyway we note that group 2 reaches its goal quicker than group 2. Looking at Fig. 8, we can check that both group are finally centered around the points $(0.65, 0.5)$ and $(0.5, 0.65)$.
Finally we remark that we obtain the opposite (or symmetric) situation, i.e. when starting to optimize with group 1. Then, the symmetry breaking seems to be a consequence of this starting choice.

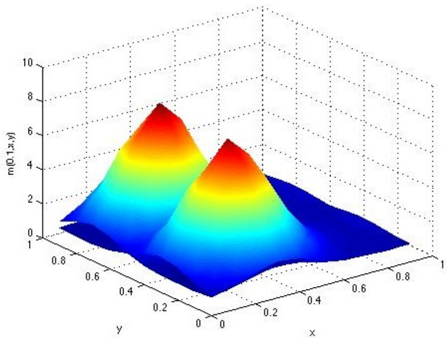


(a) Initial densities

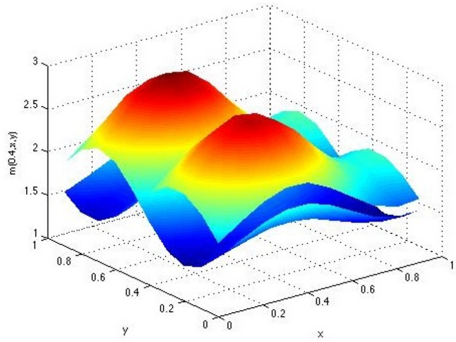


(b) Final costs Ψ^i

Figure 5: Data

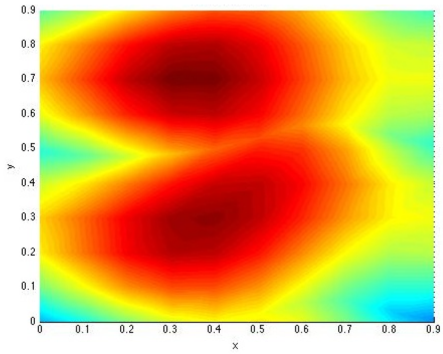


(a) Densities at instant $t = 0.1$

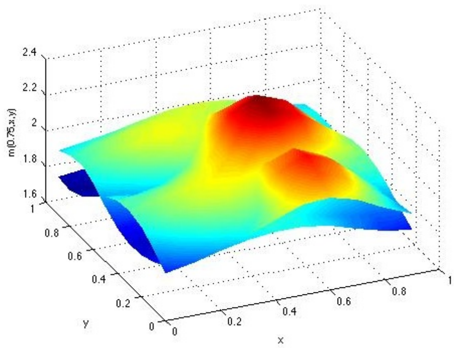


(b) Densities at instant $t = 0.4$

Figure 6: Spreading over of m^1 and m^2

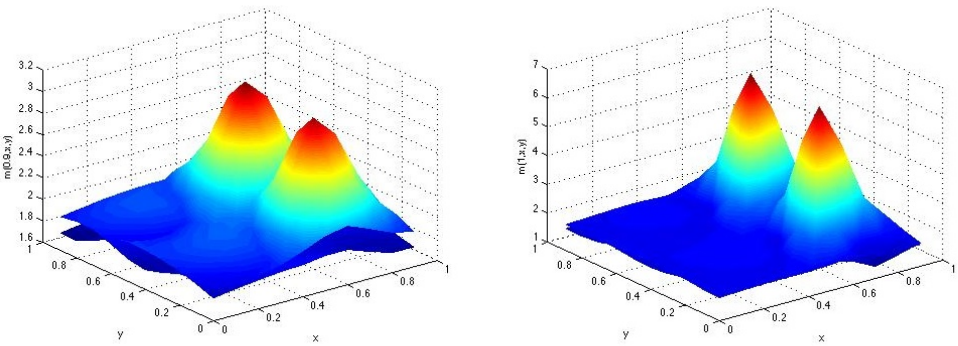


(a) Densities contour lines at instant $t = 0.65$



(b) Densities at instant $t = 0.75$

Figure 7: Group 2 go through the middle of the domain as some agents of group 1, nevertheless most of group 2 individuals transit by the border (periodic conditions)



(a) Densities at instant $t = 0.9$ (b) Densities at instant T=1

Figure 8: End of the period evolution

4. Congestion: stationary lane formation

This section is devoted to mean field game congestion models. We start with the motivation for a single population, and then present the generalization for two populations. Note that although the basic assumptions on the cost functional are quite simple, the model is capable of reproducing interesting phenomena like lane formation.

4.1. Single species congestion model

Here we define as in Lions (2008-2010) the running cost as $f_i(t, x, \alpha, m) = (\frac{|\alpha|^q}{q}(\hat{x}^{-i})^a + k(t, x^i))e^{-rt}$, where:

- k models panic (this permanent cost does not depend on density or time, the larger it is the faster people want to get out),
- $r > 0$ is the discount factor (i.e. the actual position of all other people affects the strategy of an individual much more than the evolution in time),
- we write here the general case where $q > 1$ but we will in fact focus, as usual, on the quadratic case $q = 2$
- the nonnegative power index a models how congestion impacts the cost to move. Note that the running cost increases with the a-th power of the distribution of people, forcing individuals to avoid congested areas. Note that for $a = 0$ the cost does not penalize congestion (for a fixed value of q). Furthermore we can show uniqueness of solutions if $a < a^*$, where a^* is a nonnegative constant depending on q (see Lions (2008-2010)) for more details).

Then the corresponding stationary MFG (see Lasry and Lions (2007) and Lions (2008-2010) for the detailed derivation) is given by

$$\nu\Delta u + \frac{1}{p}\frac{|\nabla u|^p}{m^b} + -ru + \lambda = k \tag{20a}$$

$$-\nu\Delta m + \mathrm{div}(m\frac{(\nabla u)^{p-1}}{m^b}) = 0 \tag{20b}$$

$$m > 0, \int u dx = 0, \quad \int m\, dx = 1, \tag{20c}$$

with $\nu = \frac{\sigma^2}{2}$, $b = \frac{a}{q-1}$ and $p = \frac{q}{q-1}$.

4.2. Two-population model

The corresponding two species model looks slightly different. Here we assume that each density would like to avoid congestion within its own group as well as with the other. The corresponding running cost is given by

$$f_i(t, x, \alpha, m) = \left(\frac{|\alpha|^q}{q}(m_i)^a (m_j)^{\tilde{a}} + k(x, t)\right) e^{-rt} \tag{21}$$

for $i = 1, 2$, with $m = (m_1, m_2)$. Note that when $\tilde{a} > a$, individuals of one crowd primarily avoid congestion with people from the other group. Then the corresponding mean field game for m_1 and m_2 reads as

$$\nu\Delta u_i + \frac{1}{p}\frac{|\nabla u_i|^p}{m_i^b m_j^{\tilde{b}}} - ru_i + \lambda_i = k \tag{22a}$$

$$-\nu\Delta m_i + \mathrm{div}(m_i \frac{(\nabla u_i)^{p-1}}{m_i^b m_j^{\tilde{b}}}) = 0 \tag{22b}$$

$$m_i > 0, \int u_i dx = 0, \quad \int m_i \, dx = 1, \tag{22c}$$

with $i, j = 1, 2$ and $\tilde{b} = \frac{\tilde{a}}{q-1}$. To avoid the division by zero we consider a slightly different model (which corresponds to consider $(c + m)$ instead of m in the running cost (21)), namely

$$-\nu\Delta u_i + \frac{1}{p}\frac{|\nabla u_i|^p}{(c + m_i)^b (c + m_j)^{\tilde{b}}} - \kappa u_i + \lambda_i = k \tag{23a}$$

$$-\nu\Delta m_i - \mathrm{div}(m_i \frac{(\nabla u_i)^{p-1}}{(c + m_i)^b (c + m_j)^{\tilde{b}}}) = 0 \tag{23b}$$

$$m_i > 0, \int u_i dx = 0, \quad \int m_i \, dx = 1. \tag{23c}$$

for a positive constant c.

We choose homogeneous Dirichlet conditions for m_i at the exit (people leave the room, hence the density has to be zero) and a homogenous Neumann boundary conditions on the rest of the boundary, i.e.

$$m_i = 0 \quad \text{for all } x \in \Gamma_i^{out} \text{ and } \frac{\partial m_i}{\partial n} = 0 \quad \text{on the rest of the boundary.} \tag{24}$$

The same boundary conditions are set for u_i. Here the additional variable λ_i and the integral condition for u_i are not necessary. We replace the integral condition for m_i by a source term in the Kolmogorov equation, i.e.

$$-\nu\Delta m_i - \mathrm{div}(m_i \frac{(\nabla u_i)^{p-1}}{(c + m_i)^b (c + m_j)^{\tilde{b}}}) = f(x).$$

This source term can be interpreted as an exit of an underground or a large supermarket.

Figure 9: Sketch of the computational domain with two exits

4.3. Numerical simulations

Finally we would like to illustrate the behavior of the proposed congestion model (23) with various numerical simulations. Note that system (23) is a coupled system of four nonlinear PDEs, which can be solved using Newtons' method. The abstract formulation of (23) is given by

$$F(u_1, m_1, u_2, m_2) = 0$$

where F is a nonlinear operator. Hence Newtons' method reads as

$$JF(w^n)q^n = -F(w^n),\ w^{n+1} = w^n + \tau q^n,$$

where $w^n = (u_1^n, m_1^n, u_2^n, m_2^n)$ denotes the n-th Newton iterate, $JF(w^n)$ is the Jacobi matrix evaluated at w^n and τ is an appropriate damping parameter. We use a hybrid discontinuous Galerkin method to discretize the Newton system in space. All computations and simulations are based on the softwares package Netgen/NgSolve (cf. Schöberl (1997)) and Pardiso (cf. Schenk and Gärtner (2004b,a)). Throughout this section we choose the following parameters

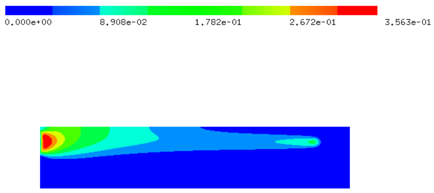
$$q = 2,\ r = 1 \text{ and } k = 1.$$

4.3.1. Validation of the model

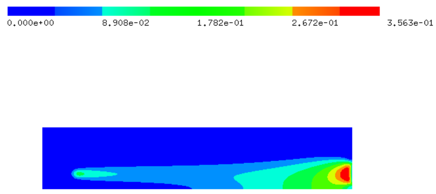
First we would like validate our mathematical model with a simple test case. We choose a rectangular domain of size $[-1, 1] \times [-0.2, 0.2]$, that has two exits (see Figure 9). We have two sources f_i, $i = 1, 2$ (Gaussians) located at $(0.8, -0.1)$ and $(-0.8, -0.1)$. The first group located at $(0.8, 0.1)$ wants to exit at the upper left door, while the other one wants to get to the lower right one. We assume that people want to avoid congestion within their own and the other group at the same level, i.e. $a = \tilde{a} = 0.5$. The diffusivity parameter is set to $\nu = 0.05$.
The intuitive solution of this problem is that each group stays in its "predefined" lane, that is determined by the offset of both sources. Our numerical simulations confirm this intuitive solution, we observe the formation of two lanes in Figure 10.

4.3.2. Avoidance behavior

Next we would like to study the avoidance behavior if two groups move towards each other without the possibility to totally avoid the other group. We choose the same domain as in the first example, only the exits are different. Here the exits are located at $x = \pm 1$. We put two sources of people at $(\pm 0.8, 0)$, each group wants to move to the opposite exit. Again we set $a = \tilde{a} = 0.5$ and $\nu = 0.05$. The solution is depicted in Figure 11, we observe that both groups move towards each other. Because of the high densities at $x = \pm 0.8$, the groups split and merge behind the high density regions.

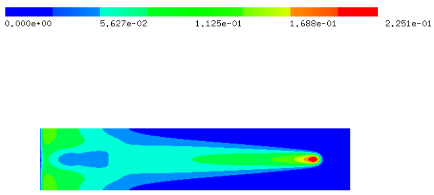


(a) Population m_1

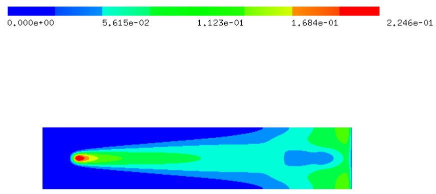


(b) Population m_2

Figure 10: Validation: top view of both groups



(a) Population m_1



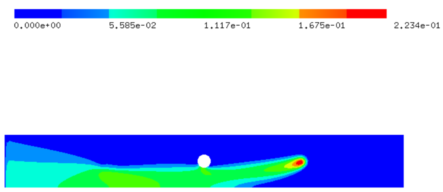
(b) Population m_2

Figure 11: Avoidance behavior: top view of both groups

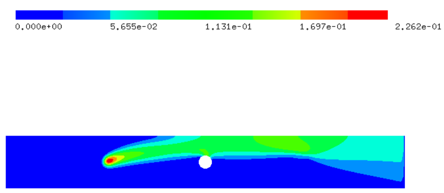
4.3.3. Obstacle problem - symmetry breaking

Let us consider a domain with a circular obstacle, i.e. a rectangle $[-1.5, 1.5] \times [-0.2, 0.2]$ with a circular obstacle of radius $r = 0.05$ located at $(0, 0)$. Here we place two sources of people at $(\pm 0.75, 0)$, the exists are located on the right and the left of the domain. We assume that the groups do not strongly mind congestion within themselves, but clearly with the other group. Hence we choose two different coefficients a and $\tilde{a}$, i.e. $a = 0.25$ and $\tilde{a} = 0.75$. Here the diffusion parameter is set to $\nu = 0.05$. The solution of the Newton iteration is illustrated in Figure 12. Here we observe the loss of symmetry and the consequent formation of lanes, i.e. each group passes the obstacle on one side.

We think that the different values of a and $\tilde{a}$ facilitate the symmetry breaking. Furthermore we suspect that the geometry of the domain as well as the diffusion parameter play another role in this complex process.

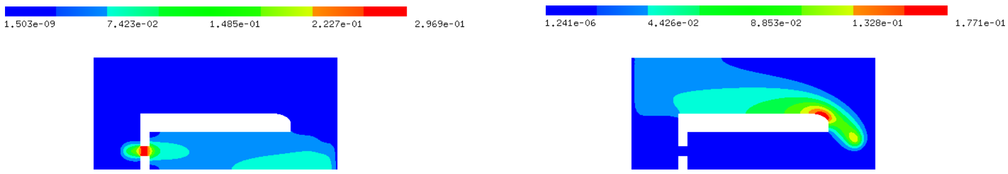


(a) Population m_1



(b) Population m_2

Figure 12: Obstacle problem: top view of both groups



(a) Population m_1

(b) Population m_2

Figure 13: Corridors: top view of both groups

4.3.4. Corridors

Finally we consider a more complicated domain, i.e. a rectangular domain which has an L-shaped obstacle in the center (see Figure 13) . This obstacle creates a small "door" on the left side. We place two sources at the lower left and lower right corner at $(0.35, 0.3)$ and $(1.4, 0.3)$. The exits are located at the right and left side of the rectangle, each group wants to get to the opposite exit. We set $a = 0.5$ and $\tilde{a} = 2$, i.e. people within a group try to avoid congestion with the other group but do not mind congestion as much within their own group. The diffusion parameter is set to $\nu = 0.1$. Again we observe lane formation, the first group takes the small door to exit at the top edge, while the second group takes the left corridor to exit on the bottom edge (see Figure 13). From our viewpoint this is a reference example where pure forward models may fail to describe this "anticipation" solution. Indeed, there is no reason why the pedestrians coming from the right source should avoid the lower corridor if they do not anticipate that people coming from the left source will reach the small door much faster (which is naturally a congested area).

5. Conclusion and outlook

In this paper we presented a new mathematical modeling approach for crowd motion models. It is based on the theory of mean field games, which assumes that individuals are smart players which try to optimize their strategy and path with respect to certain costs (equilibrium with rational expectations approach). This clearly differentiates MFG from robot or automata models. We focused on modeling aversion and congestion behavior between two groups of pedestrians via MFG. In both cases we provided numerical solutions to approximate the solutions. Here we were able to reproduce characteristic features of crowds like lane formation or symmetry breaking. We showed that many situations can be treated via MFG (finite horizon vs stationary problems, aversion vs congestion, objective function vs panic, boundary conditions and sources terms). The main feature of MFG is the forward-backward structure, the usual forward part describes the crowd evolution while the backward gives the process of how the expectations are built.

To the best of our knowledge this is one of the first applications of MFG to pedestrian crowd modeling. It is thus necessarily imperfect and many things remain to be done. Here we list some of them. First it would be interesting to introduce a common noise (e.g. modeling an earthquake incidence or a bridge failure's rumor etc.). This is feasible in a MFG setting, but leads to infinite dimensional PDEs (thus more complicated). Furthermore we would like to include the realistic assumption that people can not observe the whole distribution of the crowds. This could be accomplished by introducing partial blindness. The goal is then to model myopic individuals.

We think that the proposed MFG models give a good impression of the possibilities of mean field games theory. These models may serve as a first and promising step towards the development of more realistic MFG models for crowd motion.

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